Propagation in Arbitrarily Magnetized Ferrites Between Two Conducting Parallel Planes*

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Summary—**The electromagnetic waves propagating in arbitrarily magnetized homogeneous ferrites between two perfectly conducting parallel planes have been investigated by using the operational calculus method. The discrete propagation constants and the eigenvalues are to be determined from an algebraic equation of the fourth order and a determinantal equation derived from the boundary conditions. The hybrid modes thus found degenerate to the solutions already found for the particular cases of longitudinally and transversely (parallel and perpendicular to boundaries) magnetized ferrite cases.**

I. Introduction

IN RECENT YEARS the magnetized ferrites with their gyrotropic media characteristics have been used extensively in waveguide applications. The general mathematical analysis of electromagnetic waves in a homogeneous, gyrotropic magnetized ferrite (or plasma) in a waveguide is important and rather difficult. Two separate boundary-value problems are usually considered:

1) The longitudinally magnetized gyrotropic media—the static magnetic field is in the direction of propagation.

2) The transversely magnetized gyrotropic media—the static magnetic field is perpendicular to the direction of propagation.

Polder [1] derived the tensor permeability of the ferrite media. Hogan [2] made experimental studies of the propagation in a cylindrical waveguide containing ferrites, and of the Faraday rotation effect. Suhl and Walker [3], Gamo [4], Kales [5], and Ginzburg [6] solved the problem of a circular waveguide completely filled with a longitudinally magnetized ferrite. Van-Trier [7] discussed the modes which can exist in a parallel plane waveguide. Mikaelyan [8] used reflection of waves in order to solve the problem of propagation in a rectangular waveguide with transversely magnetized ferrite. Chambers [9] discussed the solution for a cylindrical waveguide of arbitrary cross section, filled with a longitudinally magnetized ferrite. A general theoretical approach to the problem and a discussion of previous results has been given by Epstein [10]. An extensive summary has been given by Kales [11].

General formulations of the equations and results for point sources and discontinuities in waveguides were given by Bresler [12]–[14]. Mikaelyan [8], Seidel [15], and Barzilai and Gerosa [16] used the arbitrarily directed plane wave as the basic solution in magnetized ferrites. A similar idea for a more general medium has been used by Unz [17], [18]. Sandler [19] discussed the circular-cylinder problem with longitudinal magnetization, using reflection of waves. Barzilai and Gerosa [20] solved the problem of propagation in a rectangular waveguide with longitudinally magnetized ferrite.

Bunkin [21] gave the formulation for the radiation from current sources in an anisotropic medium. This was extended later by Chow [22]. Tyras and Held [23] discussed the radiation problem from the open end of a waveguide with magnetized ferrite. Lax and Button [24] discussed nonreciprocal applications. Tonning [25] calculated the energy densities in anisotropic media. Arbel [26] discussed the radiation from a point source in a bounded anisotropic medium.

Besides the references mentioned above, there are numerous other articles discussing the propagation in gyrotropic media, the corresponding boundary-value problems, and related material. Extensive bibliography may be found in several books and reports [17], [27]–[30].

The purpose of this paper is to solve a more general problem—the propagation of electromagnetic waves in a homogeneous, arbitrarily magnetized ferrite between two perfectly conducting parallel planes. It is proposed to solve this problem by using the Fourier integral (or series) transformation and the operational calculus method. This method has been used by Unz [17], [18] in order to discuss propagation in general media and to solve propagation problems in transversely and longitudinally magnetized ferrites between two parallel perfectly conducting planes. It will be shown that our general solution will include the above-mentioned results as particular cases.

II. The Propagation Constant

The equations which govern the propagation of electromagnetic waves in a homogeneous, gyrotropic, source-free medium, assuming harmonic time variation $e^{+j\omega t}$ are as follows:

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$$\nabla \times \overline{H} = j\omega\overline{D} \tag{1a}$$

$$\nabla \times \overline{E} = -j\omega\overline{B} \tag{1b}$$

$$\overline{D} = \epsilon\overline{E} \tag{2a}$$

$$\overline{B} = (\mu)\overline{H} \tag{2b}$$

where for the case of arbitrarily magnetized ferrites [31] (μ) is a tensor of the form

$$(\mu) = \begin{bmatrix} \mu_1 & p_1 & p_2^* \\ p_1^* & \mu_2 & p_3 \\ p_2 & p_3^* & \mu_3 \end{bmatrix}. \tag{3}$$

Assuming the static magnetic field $\overline{H}_0$ in the direction (θ, ϕ) as in Fig. 1, the components of the tensor in (3) are given [31] by

$$\mu_1 = \mu + (\mu_0 - \mu)\sin^2\theta\cos^2\phi \tag{4a}$$

$$\mu_2 = \mu + (\mu_0 - \mu)\sin^2\theta\sin^2\phi \tag{4b}$$

$$\mu_3 = \mu_0 - (\mu_0 - \mu)\sin^2\theta \tag{4c}$$

$$p_1 = \tfrac{1}{2}(\mu_0 - \mu)\sin^2\theta\sin 2\phi + jk\cos\theta \tag{4d}$$

$$p_2 = \tfrac{1}{2}(\mu_0 - \mu)\sin 2\theta\cos\phi + jk\sin\theta\sin\phi \tag{4e}$$

$$p_3 = \tfrac{1}{2}(\mu_0 - \mu)\sin 2\theta\sin\phi + jk\sin\theta\cos\phi. \tag{4f}$$

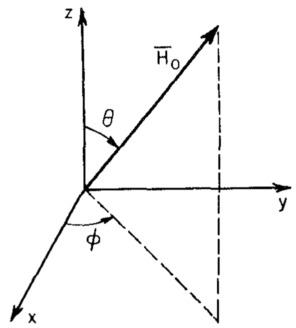

Fig. 1—The static magnetic field.

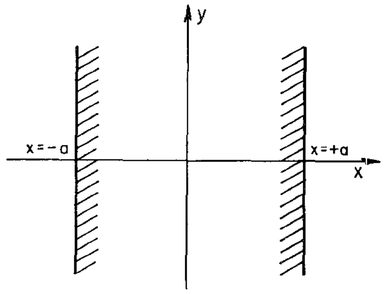

Fig. 2—Parallel-plane guide.

Substituting (2) and (3) into (1) and using (5) and similar relationships, one gets, after rearranging,

$$\omega\epsilon E^x - \gamma H^y = 0 \tag{6a}$$

$$\omega\epsilon E^y + \gamma H^x + \alpha H^z = 0 \tag{6b}$$

$$\omega\epsilon E^z - \alpha H^y = 0 \tag{6c}$$

$$\gamma E^y + \omega\mu_1 H^x + \omega p_1 H^y + \omega p_2^* H^z = 0 \tag{6d}$$

$$-\gamma E^x - \alpha E^z + \omega p_1^* H^x + \omega\mu_2 H^y + \omega p_3 H^z = 0 \tag{6e}$$

$$\alpha E^y + \omega p_2 H^x + \omega p_3^* H^y + \omega\mu_3 H^z = 0. \tag{6f}$$

p_1^*, p_2^*, p_3^* are the complex conjugate values. The tensor in (3) reduces to simpler well-known forms for particular cases. One can see that for a lossless case, (μ) in (3) in Hermitian.

Taking the two perfectly conducting parallel planes to be perpendicular to the x axis, as in Fig. 2, one could assume a solution independent of the y axis, for propagation in the z direction, in the following form:

$$E_x(x, z) = \sum_\alpha E^x(\alpha)e^{j\alpha x}e^{-j\gamma z}. \tag{5}$$

The above solution will be applicable as long as the sources will be infinite and also independent of the y direction. Similar relationships are available for the rest of the field components. E^x is called the transformed value of E_x. In (5) one has $\gamma = \gamma(\alpha)$, and the corresponding propagating wave will include the sum of all the α components, which will give the same propagation constant γ.

The complete derivation of (6) may be found in a previous paper [17]. It is obvious that in our operational calculus method $\partial/\partial x \equiv j\alpha$, $\partial/\partial z \equiv -j\gamma$.

The homogeneous set of equations (6) will have a nontrival solution if and only if the determinant of the coefficients will be zero:

$$\begin{vmatrix} \omega\epsilon & 0 & 0 & 0 & -\gamma & 0 \\ 0 & \omega\epsilon & 0 & \gamma & 0 & \alpha \\ 0 & 0 & \omega\epsilon & 0 & -\alpha & 0 \\ 0 & \gamma & 0 & \omega\mu_1 & \omega p_1 & \omega p_2^* \\ -\gamma & 0 & -\alpha & \omega p_1^* & \omega\mu_2 & \omega p_3 \\ 0 & \alpha & 0 & \omega p_2 & \omega p_3^* & \omega\mu_3 \end{vmatrix} = 0. \tag{7}$$

From the determinantal (7), one may find the following relationship:

$$\begin{aligned} &(\omega^2\epsilon\mu_1 - \gamma^2)(\omega^2\epsilon\mu_2 - \alpha^2 - \gamma^2)(\omega^2\epsilon\mu_3 - \alpha^2) \\ &\quad - \omega^4\epsilon^2|p_3|^2(\omega^2\epsilon\mu_1 - \gamma^2) - \omega^4\epsilon^2|p_1|^2(\omega^2\epsilon\mu_3 - \alpha^2) \\ &\quad - (\omega^2\epsilon\mu_2 - \alpha^2 - \gamma^2)(\omega^4\epsilon^2|p_2|^2 + \alpha^2\gamma^2 - 2\alpha\gamma\omega^2\epsilon\,\mathrm{Re}\,p_2) \\ &\quad + 2\omega^4\epsilon^2\,\mathrm{Re}\left[(\omega^2\epsilon p_2 - \alpha\gamma)p_1p_3\right] = 0 \end{aligned} \tag{8}$$

where Re means the real part of the complex number. Eq. (8) may be rewritten as follows:

$$\mu_3\gamma^4 - 2\alpha\gamma^3 \operatorname{Re} p_2 + \{\alpha^2(\mu_1+\mu_3) + \omega^2\epsilon[|p_2|^2 + |p_3|^2 - \mu_3(\mu_1+\mu_2)]\}\gamma^2 + 2\alpha[(\omega^2\epsilon\mu_2 - \alpha^2)\operatorname{Re} p_2 - \omega^2\epsilon \operatorname{Re}(p_1p_3)]\gamma + \mu_1\alpha^4 + \omega^2\epsilon\alpha^2[|p_1|^2 + |p_2|^2 - \mu_1(\mu_2+\mu_3)] + \omega^4\epsilon^2[\mu_1\mu_2\mu_3 + 2\operatorname{Re}(p_1p_2p_3) - \mu_1|p_3|^2 - \mu_3|p_1|^2 - \mu_2|p_2|^2] = 0. \tag{9}$$

Eq. (9) is an algebraic equation of the fourth order for the propagation constant $\gamma=\gamma(\alpha)$. Since the determinant (7) is Hermitian, all the coefficients of the algebraic equation (9) are real.

III. The Field Components

In this section we will find the ratio between the different transformed field components in our solution. Disregarding (6f) and rewriting the rest of (6) one obtains:

$$\begin{array}{lllllll}
\omega\epsilon E^x & & & -\gamma H^y & & = 0 & (10a)\\
& \omega\epsilon E^y & +\gamma H^x & & +\alpha H^z & = 0 & (10b)\\
& & & -\alpha H^y & & = -\omega\epsilon E^z & (10c)\\
& \gamma E^y & +\omega\mu_1 H^x & +\omega p_1 H^y & +\omega p_2{}^* H^z & = 0 & (10d)\\
-\gamma E^x & & +\omega p_1{}^* H^x & +\omega\mu_2 H^y & +\omega p_3 H^z & = \alpha E^z. & (10e)
\end{array}$$

From (10) all the transformed components can be found in terms of E^z, by solving five linear equations with five unknowns. From (10c) one obtains

$$\frac{H^y}{E^z} = \frac{1}{\eta^y} = \frac{\omega\epsilon}{\alpha}\,. \tag{11a}$$

From (11a) and (10a) one obtains

$$\frac{E^x}{E^z} = \pi^x = \frac{\gamma}{\alpha}\,. \tag{11b}$$

Substituting (11) into (10b), (10d), and (10e) one obtains

$$\omega\epsilon E^y + \gamma H^x + \alpha H^z = 0$$

$$\gamma E^y + \omega\mu_1 H^x + \omega p_2{}^* H^z = -\frac{\omega^2\epsilon p_1}{\alpha} E^z$$

$$\omega p_1{}^* H^x + \omega p_3 H^z = \frac{\gamma^2 + \alpha^2 - \omega^2\epsilon\mu_2}{\alpha} E^z.$$

These are three linear equations with three unknowns. All the transformed components E^y, H^x, H^z may be found there in terms of the transformed component E^z. Solving by determinants, one obtains:

$$\frac{E^y}{E^z} = \pi^y = \left(\frac{1}{\alpha}\right)\frac{\omega^2\epsilon(\gamma p_1p_3 - \alpha|p_1|^2) + (\gamma^2+\alpha^2-\omega^2\epsilon\mu_2)(\gamma p_2{}^* - \alpha\mu_1)}{p_3(\omega^2\epsilon\mu_1 - \gamma^2) + p_1{}^*(\alpha\gamma - \omega^2\epsilon p_2{}^*)} \tag{11c}$$

$$\frac{H^x}{E^z} = \frac{1}{\eta^x} = \left(\frac{1}{\omega\alpha}\right)\frac{-\omega^4\epsilon^2 p_1p_3 + (\gamma^2+\alpha^2-\omega^2\epsilon\mu_2)(\alpha\gamma - \omega^2\epsilon p_2{}^*)}{p_3(\omega^2\epsilon\mu_1 - \gamma^2) + p_1{}^*(\alpha\gamma - \omega^2\epsilon p_2{}^*)} \tag{11d}$$

$$\frac{H^z}{E^z} = \frac{1}{\eta^z} = \left(\frac{1}{\omega\alpha}\right)\frac{\omega^4\epsilon^2|p_1|^2 - (\gamma^2+\alpha^2-\omega^2\epsilon\mu_2)(\gamma^2 - \omega^2\epsilon\mu_1)}{p_3(\omega^2\epsilon\mu_1 - \gamma^2) + p_1{}^*(\alpha\gamma - \omega^2\epsilon p_2{}^*)}\,. \tag{11e}$$

Eqs. (11) give the ratios of the transformed components. The factors π and η are complex and are defined above. Since the medium is homogeneous, all media parameters are constant. However, there are usually different values of α denoted by α_1, α_2, α_3, α_4. The corresponding constants in (11) for different α will be

$$\pi^x(\alpha=\alpha_1) = \pi_1{}^x \qquad \pi^x(\alpha=\alpha_2) = \pi_2{}^x$$
$$\pi^x(\alpha=\alpha_3) = \pi_3{}^x \qquad \pi^x(\alpha=\alpha_4) = \pi_4{}^x \tag{12}$$

and so on for the rest of the constants.

IV. The Boundary Conditions

In Fig. 2 the necessary boundary conditions for the parallel perfectly conducting infinite planes are

$$E_z = 0 \qquad x = -a, \quad x = +a \tag{13a}$$

$$E_y = 0 \qquad x = -a, \quad x = +a. \tag{13b}$$

From (8) or (9) one can see that for a given value of the propagation constant γ, one will get a fourth-order algebraic equation for α, which has for our problem four distinct solutions, α_1, α_2, α_3, α_4. Therefore, for a given propagation constant γ, one may write the most general solution in the form of (5), assuming the propagation factor $e^{+j(\omega t-\gamma z)}$,

$$E_z = \sum_{m=1}^{4} (A_m + jB_m)e^{j\alpha_m x} \tag{14a}$$

where A_m, B_m are real constants, to be evaluated. Using (11c) and the notation in (12), one may find from (14a)

$$E_y = \sum_{m=1}^{4} (A_m + jB_m)\pi_m{}^y e^{j\alpha_m x} \tag{14b}$$

where $\pi_m{}^y$ are complex in general. Substituting the solution (14) in the boundary conditions (13) one obtains

$$\sum_{m=1}^{4} (A_m + jB_m)e^{-j\alpha_m a} = 0 \tag{15a}$$

$$\sum_{m=1}^{4} (A_m + jB_m)e^{+j\alpha_m a} = 0 \tag{15b}$$

$$\sum_{m=1}^{4} (A_m + jB_m)\pi_m{}^y e^{-j\alpha_m a} = 0 \tag{15c}$$

$$\sum_{m=1}^{4} (A_m + jB_m)\pi_m{}^y e^{+j\alpha_m a} = 0. \tag{15d}$$

Each one of the above equations consists of two parts, real and imaginary, and each part is equal to zero. By adding (15a) and (15b) and separating the real and the imaginary parts, one may obtain

$$\sum_{m=1}^{4} A_m \cos\alpha_m a = 0 \tag{16a}$$

$$\sum_{m=1}^{4} B_m \cos\alpha_m a = 0. \tag{16b}$$

By subtracting (15b) from (15a) and separating one obtains

$$\sum_{m=1}^{4} A_m \sin\alpha_m a = 0 \tag{16c}$$

$$\sum_{m=1}^{4} B_m \sin\alpha_m a = 0. \tag{16d}$$

Taking $\pi_m{}^y = \pi_{mR}{}^y + j\pi_{m_j}{}^y$ one obtains, by adding (15c) and (15d),

$$\sum_{m=1}^{4} \left[A_m\pi_{mR}{}^y - B_m\pi_{m_j}{}^y\right]\cos\alpha_m a = 0 \tag{16e}$$

$$\sum_{m=1}^{4} \left[A_m\pi_{m_j}{}^y + B_m\pi_{mR}{}^y\right]\cos\alpha_m a = 0; \tag{16f}$$

and by subtracting (15d) from (15c),

$$\sum_{m=1}^{4} \left[A_m\pi_{mR}{}^y - B_m\pi_{m_j}{}^y\right]\sin\alpha_m a = 0 \tag{16g}$$

$$\sum_{m=1}^{4} \left[A_m\pi_{m_j}{}^y + B_m\pi_{mR}{}^y\right]\sin\alpha_m a = 0. \tag{16h}$$

In all the above calculations α_m has been assumed to be real or purely imaginary. In case α_m is a complex number, the procedure above should be modified, in order to avoid a complex determinantal equation. In this case $\alpha_m = \alpha_{m_R} + j\alpha_{m_j}$ should be substituted into (15) and the exponential terms should be rewritten explicitly into real and imaginary parts. Taking in each of (15) separately the real and the imaginary parts to be zero, one will get eight homogeneous equations with eight unknowns (A_m, B_m) with trigonometrical and exponential coefficients, which will be equivalent to the results in (16).

Eqs. (16) represent eight homogeneous linear equations with eight unknowns. One may obtain a nontrivial solution if the determinant of the coefficients of the unknowns A_m, B_m is zero:

$$\begin{array}{cccccccc} A_1 & A_2 & A_3 & A_4 & B_1 & B_2 & B_3 & B_4 \end{array}$$

$$\begin{vmatrix}
\cos\alpha_1 a & \cos\alpha_2 a & \cos\alpha_3 a & \cos\alpha_4 a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\alpha_1 a & \cos\alpha_2 a & \cos\alpha_3 a & \cos\alpha_4 a \\
\sin\alpha_1 a & \sin\alpha_2 a & \sin\alpha_3 a & \sin\alpha_4 a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin\alpha_1 a & \sin\alpha_2 a & \sin\alpha_3 a & \sin\alpha_4 a \\
\pi_{1R}{}^y\cos\alpha_1 a & \pi_{2R}{}^y\cos\alpha_2 a & \pi_{3R}{}^y\cos\alpha_3 a & \pi_{4R}{}^y\cos\alpha_4 a & -\pi_{1_j}{}^y\cos\alpha_1 a & -\pi_{2_j}{}^y\cos\alpha_2 a & -\pi_{3_j}{}^y\cos\alpha_3 a & -\pi_{4_j}{}^y\cos\alpha_4 a \\
\pi_{1_j}{}^y\cos\alpha_1 a & \pi_{2_j}{}^y\cos\alpha_2 a & \pi_{3_j}{}^y\cos\alpha_3 a & \pi_{4_j}{}^y\cos\alpha_4 a & \pi_{1R}{}^y\cos\alpha_1 a & \pi_{2R}{}^y\cos\alpha_2 a & \pi_{3R}{}^y\cos\alpha_3 a & \pi_{4R}{}^y\cos\alpha_4 a \\
\pi_{1R}{}^y\sin\alpha_1 a & \pi_{2R}{}^y\sin\alpha_2 a & \pi_{3R}{}^y\sin\alpha_3 a & \pi_{4R}{}^y\sin\alpha_4 a & -\pi_{1_j}{}^y\sin\alpha_1 a & -\pi_{2_j}{}^y\sin\alpha_2 a & -\pi_{3_j}{}^y\sin\alpha_3 a & -\pi_{4_j}{}^y\sin\alpha_4 a \\
\pi_{1_j}{}^y\sin\alpha_1 a & \pi_{2_j}{}^y\sin\alpha_2 a & \pi_{3_j}{}^y\sin\alpha_3 a & \pi_{4_j}{}^y\sin\alpha_4 a & \pi_{1R}{}^y\sin\alpha_1 a & \pi_{2R}{}^y\sin\alpha_2 a & \pi_{3R}{}^y\sin\alpha_3 a & \pi_{4R}{}^y\sin\alpha_4 a
\end{vmatrix} = 0 \tag{17a}$$

The determinantal equation (17a) may be rewritten in a symbolic form:

$$\begin{array}{cc} \text{4 terms} & \text{4 terms} \\ m = 1,2,3,4 & m = 1,2,3,4 \end{array}$$

$$\begin{vmatrix}
\cos\alpha_m a & 0 \\
0 & \cos a_m\alpha \\
\sin\alpha_m a & 0 \\
0 & \sin\alpha_m a \\
\pi_{mR}{}^y\cos\alpha_m a & -\pi_{m_j}{}^y\cos\alpha_m a \\
\pi_{mj}{}^y\cos\alpha_m a & \pi_{mR}{}^y\cos\alpha_m a \\
\pi_{mR}{}^y\sin\alpha_m a & -\pi_{mj}{}^y\sin\alpha_m a \\
\pi_{mj}{}^y\sin\alpha_m a & \pi_{mR}{}^y\sin\alpha_m a
\end{vmatrix} = 0. \tag{17b}$$

The determinantal equation (17) gives a relationship between the four eigenvalues α_1, α_2, α_3, α_4, the media parameters, and the propagation constant γ, which is introduced via the constant π^y given in (11c).

Once (17) has been found as a relationship between the eigenvalues α_m and the propagation constant γ, one of (16), for example (16h), could be disregarded, and there will remain only seven equations (16). From those seven linear equations one can find A_2/A_1, A_3/A_1, A_4/A_1, B_1/A_1, B_2/A_1, B_3/A_1, B_4/A_1 by using determinants. The results may be substituted into (14) and similar relations found from (11) in order to give the electromagnetic hybrid modes of propagation, as it was done previously [17], [18].

V. The Hybrid Modes

From (11) and (14) one may write the solution of the boundary-value problem in the form

$$E_z = \sum_{m=1}^{4} (A_m + jB_m)e^{j\alpha_m x} \tag{18a}$$

$$E_x = \sum_{m=1}^{4} (A_m + jB_m)\pi_m^x e^{j\alpha_m x} \tag{18b}$$

$$E_y = \sum_{m=1}^{4} (A_m + jB_m)\pi_m^y e^{j\alpha_m x} \tag{18c}$$

$$H_x = \sum_{m=1}^{4} (A_m + jB_m)\frac{1}{\eta_m^x} e^{j\alpha_m x} \tag{18d}$$

$$H_y = \sum_{m=1}^{4} (A_m + jB_m)\frac{1}{\eta_m^y} e^{j\alpha_m x} \tag{18e}$$

$$H_z = \sum_{m=1}^{4} (A_m + jB_m)\frac{1}{\eta_m^z} e^{j\alpha_m x} \tag{18f}$$

where the propagation coefficient $e^{j(\omega t-\gamma z)}$ is understood.

The first step in our solution is to find the propagation constant γ and the eigenvalues α_m. This can be done by the simultaneous solution of the algebraic equation (8) and the determinantal equation (17). The procedure of solution will involve numerical solutions of trial and error. For a set of propagation constants γ one could find by using (8) or (9) the corresponding eigenvalues α_1, α_2, α_3, α_4. Then one should try which ones of the solutions will obey the determinantal equation (17). Only a discrete number of values of the propagation constant γ will obey it, and it will represent the different modes of propagation. Of course these calculations will usually require extensive numerical work and the use of a computer.

Once the propagation constant γ and the eigenvalues α_m have been found for a particular mode, the coefficients A_m, B_m may be found in terms of A_1, for example, by using (15) and determinantal solutions, as described in the previous section. Substituting those coefficients in (18) and using the definitions in (11), one thus finds the propagating electromagnetic wave which will consist of a set of hybrid modes [32].

While the process of solution has been described in detail, it will involve in general quite a lot of numerical and computer work—the solutions of fourth-order algebraic equations and the evaluations of 8×8 determinants. Some examples for particular cases are given in Section VI.

VI. Particular Cases

Several particular cases of the previous general solution will be discussed here.

A. Longitudinal Magnetization

In the case of a longitudinally magnetized ferrite, the static magnetic field $\overline{H}_0$ is in the direction of propagation z and one obtains $\theta=0$ in Fig. 1. Eqs. (4) then become

$$\mu_1 = \mu_2 = \mu \qquad \mu_3 = \mu_0 \tag{19a}$$

$$p_1 = jk \qquad p_2 = p_3 = 0. \tag{19b}$$

Substituting (19) into (9) one obtains

$$\gamma^4 - \gamma^2\left[2\omega^2\epsilon\mu - \alpha^2\left(1 + \frac{\mu}{\mu_0}\right)\right] + \frac{\mu}{\mu_0}(\omega^2\epsilon\mu_0 - \alpha^2)\left(\omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2\right). \tag{20}$$

Substituting (19) into (11) one obtains

$$\frac{E^x}{E^z} = \pi^x = \frac{\gamma}{\alpha} \tag{21a}$$

$$\frac{E^y}{E^z} = \pi^y = \frac{-j}{\gamma\alpha}\frac{\mu}{k}\left(\gamma^2 + \alpha^2 - \omega^2\epsilon\frac{\mu^2 - k^2}{\mu}\right) \tag{21b}$$

$$\frac{H^x}{E^z} = \frac{1}{\eta^x} = +\frac{j}{k\omega\alpha}(\gamma^2 + \alpha^2 - \omega^2\epsilon\mu) \tag{21c}$$

$$\frac{H^y}{E^z} = \frac{1}{\eta^y} = \frac{\omega\epsilon}{\alpha} \tag{21d}$$

$$\frac{H^z}{E^z} = \frac{1}{\eta^z} = \frac{-j}{k\omega\gamma\alpha^2}\left[(\gamma^2 - \omega^2\epsilon\mu)(\gamma^2 + \alpha^2 - \omega^2\epsilon\mu) - \omega^4\epsilon^2k^2\right]. \tag{21e}$$

Since from (21b) one obtains $\pi_R^y=0$ for $\alpha=$real and also from (20) one sees that the eigenvalues α_m appear in pairs, *i.e.*,

$$\alpha_1 = \hat{\alpha}_1; \quad \alpha_2 = -\hat{\alpha}_1; \quad \alpha_3 = \hat{\alpha}_2; \quad \alpha_4 = -\hat{\alpha}_2; \tag{22a}$$

therefore, from (21b) and (22a) one obtains:

$$\pi_{ij}^y = \pi_1; \quad \pi_{2j}^y = -\pi_1; \quad \pi_{3j}^y = \pi_2; \quad \pi_{4j}^y = -\pi_2. \tag{22b}$$

Substituting (22a) and (22b) in the determinantal equation (17) one obtains finally two independent results:

$$\pi_1 \operatorname{tg} \hat{\alpha}_1 a = \pi_2 \operatorname{tg} \hat{\alpha}_2 a \tag{23a}$$

$$\frac{\operatorname{tg} \hat{\alpha}_1 a}{\pi_1} = \frac{\operatorname{tg} \hat{\alpha}_2 a}{\pi_2}\,. \tag{23b}$$

The preceding results are identical with the one derived previously, and the complete solution may be found elsewhere [18].

B. Transverse Magnetization-Parallel

In the case of a transversely magnetized ferrite, with the static magnetic field parallel to the two perfectly conducting parallel planes, *i.e.*, the static magnetic field $\overline{H}_0$ in the y axis direction, one has $\theta=90°$ and $\phi=90°$. Eqs. (4) then become

$$\mu_1 = \mu_3 = \mu \qquad \mu_2 = \mu_0 \tag{24a}$$

$$p_2 = jk \qquad p_1 = p_3 = 0. \tag{24b}$$

Substituting (24) into (9) one obtains

$$[\gamma^2 - (\omega^2\epsilon\mu_0 - \alpha^2)]\left[\gamma^2 - \left(\omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2\right)\right] = 0. \tag{25}$$

One obtains in this case two independent waves:

$$\gamma^2 = \omega^2\epsilon\mu_0 - \alpha^2 \qquad \text{TM wave} \tag{26a}$$

$$\gamma^2 = \omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2 \qquad \text{TE wave.} \tag{26b}$$

Substituting (24) into (11) one obtains the components of the two independent waves. For the TM wave,

$$\frac{E^x}{H^y} = \frac{\gamma}{\omega\epsilon}; \qquad \frac{E^z}{H^y} = \frac{\alpha}{\omega\epsilon}; \tag{27a}$$

and for the TE wave, again using (26b),

$$\frac{H^x}{E^y} = -\frac{\gamma\mu + j\alpha k}{\omega(\mu^2 - k^2)} \tag{27b}$$

$$\frac{H^z}{E^y} = j\frac{\gamma k + j\alpha\mu}{\omega(\mu^2 - k^2)}. \tag{27c}$$

In this case the solution degenerates to the point where $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha=m\pi/a$ where m=integer. The above results are identical with the ones derived previously [7], [17], and the complete solution may be found elsewhere [17].

C. Transverse Magnetization-Perpendicular

In the case of a transversely magnetized ferrite, with the static magnetic field perpendicular to the two perfectly conducting parallel planes—*i.e.*, the static magnetic field H_0 is in the x axis direction—one has: $\theta=90°$ and $\phi=0°$. Eqs. (4) then become

$$\mu_2 = \mu_3 = \mu \qquad \mu_1 = \mu_0 \tag{28a}$$

$$p_3 = jk \qquad p_1 = p_2 = 0. \tag{28b}$$

Substituting (28) into (9) one obtains

$$\gamma^4 - \gamma^2\left[\omega^2\epsilon\mu_0 + \omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2\left(1 + \frac{\mu_0}{\mu}\right)\right] + \omega^2\epsilon\mu_0\left(\omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2\right) - \frac{\mu_0}{\mu}\alpha^2(\omega^2\epsilon\mu - \alpha^2) = 0. \tag{29}$$

After some involved algebra (29) can be solved to give

$$\gamma_{1,2}^2 = \frac{1}{2}\left[\omega^2\epsilon\mu_0 + \omega^2\epsilon\frac{\mu^2 - k^2}{\mu} - \alpha^2\left(1 + \frac{\mu_0}{\mu}\right)\right] \pm \frac{1}{2}\left\{\left[\omega^2\epsilon\mu_0 - \omega^2\epsilon\frac{\mu^2 - k^2}{\mu} + \alpha^2\left(1 - \frac{\mu_0}{\mu}\right)\right]^2 + \left[2\alpha\omega^2\epsilon\mu_0\frac{k}{\mu}\right]^2\right\}^{1/2}. \tag{30}$$

Substituting (28) in (11) one obtains the components of the waves as follows:

$$\frac{E^x}{E^z} = \pi^x = \frac{\gamma}{\alpha} \tag{31a}$$

$$\frac{E^y}{E^z} = \pi^y = j\frac{\mu_0}{k}\frac{\gamma^2 + \alpha^2 - \omega^2\epsilon\mu}{\omega^2\epsilon\mu_0 - \gamma^2} \tag{31b}$$

$$\frac{H^x}{E^z} = \frac{1}{\eta^x} = -j\frac{\gamma}{\omega k}\frac{\gamma^2 + \alpha^2 - \omega^2\epsilon\mu}{\omega^2\epsilon\mu_0 - \gamma^2} \tag{31c}$$

$$\frac{H^y}{E^z} = \frac{1}{\eta^y} = \frac{\omega\epsilon}{\alpha} \tag{31d}$$

$$\frac{H^z}{E^z} = \frac{1}{\eta^z} = -\frac{j}{\alpha\omega k}[\gamma^2 + \alpha^2 - \omega^2\epsilon\mu]. \tag{31e}$$

In this case also the solution degenerates to the point where $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha=m\pi/a$ where m=an integer. The above results are identical with the ones derived previously [8], and the complete solution may be found elsewhere [17].

Conclusion

The problem of electromagnetic wave propagation in arbitrarily magnetized ferrites between two perfectly conducting parallel planes has been solved by using an operational calculus method. A general algebraic equation of the fourth order, found from the Maxwell's equations and the media constitutive relations, gives the relationship between the propagation constant γ and the eigenvalues α. A transcendental equation, in the form of an 8×8 determinantal equation found from the boundary conditions, gives an additional relationship. By numerical methods those two equations may be solved. In general there will be four distinct, unidirectional waves for each mode of propagation.

Once the discrete propagation constants have been calculated, the corresponding field components can be evaluated by using the relationships which have been found. It has been shown that each hybrid mode of propagation degenerates to previously found [17], [18] particular cases when we take the static magnetic field in one of the following directions:

1) Direction of Propagation (longitudinally magnetized).
2) Perpendicular to direction of propagation (transversely magnetized) and parallel to boundaries.

3) Perpendicular to direction of propagation (transversely magnetized) and perpendicular to boundaries.

The above solution for the arbitrarily magnetized ferrites can be used in order to find the corresponding solution for the arbitrarily magnetized plasma, by using certain transformation ideas given previously [17] while keeping the boundary conditions invariant. The present solution can be also used in order to solve the propagation of electromagnetic waves in arbitrarily magnetized ferrites and plasma in a rectangular waveguide, by using ideas similar to ones presented by Mikaelyan [8] and by Barzilai and Gerosa [16] in their solutions.

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